JOURNAL OF COMBINATORIAL THEORY, Series A 38, 91–93 (1985)

Note

On Extendable Planes of Order Ten

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Communicated by Marshall Hall, Jr.

Received January 24, 1983

It is well known that the only way of extending a projective plane of order n (conceivable orders are 2, 4 and 10) is adjoining a set of hyperovals to the given projective plane. A converse is proved in this note. It is shown as a corollary that the existence of an extendable plane of order 10 is equivalent to the existence of a quasi-symmetric 2-(111, 12, 10)-design. © 1985 Academic Press, Inc.

INTRODUCTION

In this note we consider a (point) extension of a projective plane of order n to a 3-$(n^2+n+2, n+2, 1)$-design. Suppose π is a projective plane of order n and D is its extension where $D_\infty = \pi$ (see [1] for notations). Then the following result is well known. Its first part was proved by Hughes [4] and the remaining parts are easy verifications.

PROPOSITION 1. *With everything as above*:

(a) $n = 2, 4, 10$.

(b) *The blocks of D not containing ∞ form a class C of hyperovals of π with the property that any two hyperovals of C have* 0 *or* 2 *points in common.*

(c) *Let L be the incidence structure whose blocks are hyperovals in C. Then L is a quasi-symmetric 2-$(n^2+n+1, n+2, n)$-design (see* [3, *p.* 25] *for definition).*

We prove the following:

PROPOSITION 2. *Let L be a quasi-symmetric 2-$(n^2+n+1, n+2, n)$-design. Then there exists a unique projective plane π of order n such that the*

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blocks of L form a class of hyperovals of π and the lines of π augmented by a new point together with all the blocks of L give extension of π.

COROLLARY 3. *A projective plane of order* 10 *has an extension if and only if there exists a quasi-symmetric* 2-(111, 12, 10)*-design.*

It will follow from the proof of Proposition 2 that the condition of being "quasi-symmetric" on L in the statement of Proposition 2 can be replaced by the following: no two blocks of L intersect in one point.

Proof of Proposition 2. The divisibility conditions given $n = 2$, 4 or 10 as in [4]. Since the other cases are well known and easy we might as well take $n = 10$. Then L is a quasi-symmetric 2-design with $v = 111$, $k = 12$, $\lambda = 10$, $b = 925$ and $r = 100$.

Claim I. The block intersection numbers in L are 0 and 2.

Proof. Let there be m blocks meeting a given block B in x points and $(924 - m)$ which meet B in y points, $0 \leqslant x < y$. We have

$$\begin{aligned} mx + (924 - m)y &= 12.99 \\ m\binom{x}{2} + (924 - m)\binom{y}{2} &= \binom{12}{2}.99. \end{aligned} \tag{$*$}$$

If $x = 1$ we can solve and find $y = 4\frac{1}{2}$, an absurdity. So $x \neq 1$. Also $y = 1$ and hence $x = 0$ is impossible. From $(*)$ we have

$$mx(x-2) + (924 - m)y(y-2) = 0$$

and since $x \neq 1$, $y \neq 1$ both the summands on the left are non-negative and therefore $x = 0$, $y = 2$ and $m = 330$.

For any pair x, y of distinct points of L, define

$$S_{xy} = \{z\colon z = x \text{ or } z = y \text{ or } z \text{ is not on any block containing both } x \text{ and } y\}.$$

From Claim I it follows that any two blocks containing x and y do not have any other point in common. Hence $|S_{xy}| = v - \lambda(k-2) = 11$.

Claim II. Suppose x, y, z are three distinct points with $z \in S_{xy}$. Then $S_{xy} = S_{xz}$.

Proof. Let B_i, C_j, $i, j = 1, 2, \ldots, n$, be, respectively, the blocks containing the pair (x, y) and the pair (x, z). Clearly $B_i \neq C_j$ for otherwise $z \in B_i$, a contradiction to $z \in S_{xy}$. Hence by Claim I, $|B_i \cap C_j| = 2$. Let $T = \{x, y, z\}$ and write $\bar{B}_i = B_i \backslash T$, $\bar{C}_j = C_j \backslash T$. Then $|\bar{B}_i \cap \bar{C}_j| = 1$. Again by Claim I, $\bar{C}_j$ are pairwise disjoint and therefore $|\bar{B}_i \cap (\bigcup \bar{C}_j)| = 10 = |\bar{B}_i|$ which implies $\bar{B}_i$

and hence $\bigcup \bar{B}_i$ is contained in $\bigcup \bar{C}_j$. Cardinality considerations now force $\bigcup \bar{B}_i = \bigcup \bar{C}_j$. Hence $S_{xy} = S_{xz}$.

Claim III. Suppose $z, w \in S_{xy}$ with $z \neq w$. Then $S_{xy} = S_{zw}$.

Proof. Assume that x, y, z, w are all distinct; the proof in the remaining cases is similar. By Claim II, $S_{xy} = S_{xz}$ implying $w \in S_{xz}$ which by Claim II again yields $S_{xz} = S_{zw}$. Hence $S_{xy} = S_{zw}$.

Define an incidence structure π on the points of L as follows. The lines of π are just all the distinct sets S_{xy} and the incidence defined by inclusion. For any pair x, y of distinct points, S_{xy} is a line of π containing them both and this line is unique by Claim III. Hence π is a 2-(111, 11, 1)-design, i.e., a projective plane of order 10.

All the remaining assertions are easy to prove except the "uniqueness" assertion. To this end we have:

Claim IV. Let, for $i = 1, 2$, π_i be a projective plane of order n, L_i a class of hyperovals of π_i and D_i a 3-$(n^2 + n + 2, n + 2, 1)$-design obtained by extending π_i using the class of hyperovals L_i. Suppose L_1 and L_2 are isomorphic. Then π_1 and π_2 are isomorphic.

Proof. Let α be the (incidence preserving) isomorphism between the two quasi-symmetric designs L_1 and L_2. Then α induces a bijection from the point-set of π_1 to the point-set of π_2 since the point-set of π_i is the same as that of L_i. To show that α gives an isomorphism from π_1 to π_2, therefore, it suffices to prove that α preserves collinearity. Let x_1, y_1, z_1 be three distinct points of π_1 whose images in π_2 are x_2, y_2, z_2. Suppose x_1, y_1, z_1 are collinear in π_1. We claim that x_2, y_2, z_2 are also collinear in π_2. Suppose not. Since D_2 is a 3-design with $\lambda = 1$ there is a unique hyperoval in L_2 containing x_2, y_2, z_2. Applying α^{-1} to L_2 we find that x_1, y_1, z_1 are contained in some hyperoval in L_1, a contradiction since D_1 is a 3-design with $\lambda = 1$. Hence x_2, y_2, z_2 are collinear in π_2. Reversing the argument it is easy to see that x_1, y_1, z_1 are collinear if and only if x_2, y_2, z_2 are collinear. Hence α induces an isomorphism from π_1 to π_2.

References

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